

A scenario for critical scalar field collapse in AdS_3

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Abstract

We present a family of exact solutions, depending on two parameters α and b (related to the scalar field strength), to the three-dimensional Einstein-scalar field equations with negative cosmological constant Λ . For $b = 0$ these solutions reduce to the static BTZ family of vacuum solutions, with mass $M = -\alpha$. For $b \neq 0$, the solutions become dynamical and develop a strong spacelike central singularity. The $\alpha < 0$ solutions are black-hole like, with a global structure topologically similar to that of the BTZ black holes, and a finite effective mass. We show that the near-singularity behavior of the solutions with $\alpha > 0$ agrees qualitatively with that observed in numerical simulations of subcritical collapse, including the independence of the near-critical regime on the angle deficit of the spacetime. We analyze in the $\Lambda = 0$ approximation the linear perturbations of the self-similar threshold solution, $\alpha = 0$, and find that it has only one unstable growing mode, which qualifies it as a candidate critical solution for scalar field collapse.

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1 Introduction

In 1993 Choptuik [1] studied the spherically symmetric collapse of a massless scalar field in four-dimensional Einstein gravity and found numerical evidence of a critical behaviour at the threshold of black hole formation. The black hole threshold is reached when a certain parameter p characterizing the initial data reaches a critical value p_* . The critical behavior, since then observed in other systems and various spacetime dimensions (for a review, see [2]), is characterized by the emergence of a (continuous or discrete) self-similarity and a power-law scaling of the black hole mass $M \sim (p - p_*)^{s\gamma}$ where γ is a universal exponent, and s depends on the dimension. Inspired by this work, several years later Pretorius and Choptuik [3] and Husain and Olivier [4] performed numerical simulations of circularly symmetric gravitational collapse of a massless scalar field minimally coupled to $2 + 1$ dimensional Einstein gravity with a negative cosmological constant Λ . These simulations showed the appearance of a critical regime near the threshold of black hole formation, with continuous self-similarity and power-law scaling. A class of exact continuously self-similar solutions of the $\Lambda = 0$ equations, depending on an integer n , was constructed by Garfinkle [5], who found that the $n = 4$ solution was a good fit to the numerical data of [3] for critical collapse near the singularity (where the effect of the cosmological constant is negligible). The linear perturbation analysis of the Garfinkle family of solutions, performed in [6], showed that with suitable boundary conditions the $n = 2$ Garfinkle solution admitted a single growing mode, suggesting that this solution should be the critical one near the singularity. In [7], the Garfinkle solutions were extended to solutions of the full field equations truncated to first order in the cosmological constant Λ , and the zeroth order linear perturbation analysis of [6] was extended to the same order, confirming the results of that analysis.

However, several points remained obscure. Is there really a good reason to prefer the $n = 2$ solution over the $n = 4$ one, which fits better the observations [6, 2]? Also, the knowledge of the critical solution in the zeroth order, or even in the first order, approximation sheds no light on the global structure of the exact critical solution. What we feel is perhaps a more important drawback of the Garfinkle approach is its inability to explain a peculiar feature uncovered in the analysis of [3]. They observed that the introduction of a point particle, in other words a central conical singularity, left unchanged (up to a phase shift in proper time) the critical solution, even for an angle deficit very close to 2π . This suggests an alternative scenario in which the critical solution, instead of having a regular timelike center

as assumed in [5], should more probably have no timelike center at all, in close analogy with the case of the BTZ vacuum (or extreme) solution, which lies at the threshold between BTZ black holes with positive mass M and no center, and AdS spacetime, generically with a conical singularity (unless $M = -1$).

Recently, an analytical solution to the Einstein-scalar field equations with $\Lambda < 0$ was derived from a self-similar ansatz, and argued to be relevant to critical collapse [8]. However this argument was shown in [9] to be incorrect as presented. It is nevertheless suggestive, in view of the above remarks, that this solution is conformal to the three-dimensional Minkowski cylinder (Minkowski space with one of the cartesian spatial coordinates periodically identified), and thus has no center. The purpose of the present paper is to further explore the properties of this solution and those of a class of neighboring exact solutions. We shall show that, while a number of these properties agree qualitatively with those expected for the critical solution, the quantitative comparison with the numerical simulations shows that it cannot be the correct critical solution. The shortcomings of this solution are probably due to its lack of an asymptotic AdS region, presumably a direct consequence of its being a separable solution. Nevertheless, we believe that our analysis points the way towards a possible exact critical solution within the no-center scenario for scalar field collapse in AdS_3 .

In Sect. 2 of this paper, after briefly recalling the separable solutions constructed in [10], we focus on a class of solutions depending on two parameters b (the scalar field strength) and α real. We first show that the vacuum solutions ($b = 0$) of this class correspond respectively to sectors of the BTZ black hole spacetime with mass $M = -\alpha$ for $\alpha < 0$, of the BTZ vacuum for $\alpha = 0$, and of the AdS spacetime with a conical singularity for $\alpha > 0$. The solutions with $b^2 \neq 0$ have in common a strong spacelike central singularity, but differ by their global structure. We show that the $\alpha = 0$ solution of [8] lies at the threshold between dynamical black-hole like solutions for $\alpha < 0$ and solutions with a naked timelike central conical singularity for $\alpha > 0$ (or a regular timelike center for $\alpha = 1$). We then consider in Sect. 3 the limit $\Lambda \rightarrow 0$, which is relevant to the behavior of the corresponding $\Lambda < 0$ solutions near the spacelike singularity. The behavior of the solutions below the threshold ($\alpha > 0$) agrees qualitatively with the results of the numerical simulations of [3]. Finally we address in Sect. 4 the crucial issue of the linear perturbations of the $\alpha = 0$ solution. Enforcing suitable boundary conditions, we find only one growing mode, corresponding to the small α solution, as required for the critical solution. In the closing section we summarize our results and discuss the value of the critical exponent.

2 A class of separable solutions

The Einstein-scalar field equations are

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa \left[\partial_\mu \phi \partial_\nu \phi - \frac{1}{2}g_{\mu\nu} \partial^\lambda \phi \partial_\lambda \phi \right], \quad D^\lambda \partial_\lambda \phi = 0, \quad (2.1)$$

where Λ is the cosmological constant, and κ the Einstein gravitational constant, set to one in the following. It was shown in [10] that, in three space-time dimensions, the ansatz

$$ds^2 = F^2(T) \left[-dT^2 + dR^2 + G^2(R) d\theta^2 \right], \quad \phi = \phi(T) \quad (2.2)$$

leads to a solution of the field equations, provided the functions $G(R)$, $F(T)$ and $\phi(T)$ solve the differential equations

$$G'^2 - \gamma G^2 = \alpha, \quad (2.3)$$

$$\dot{F}^2 - \gamma F^2 = \Lambda F^4 + b^2, \quad (2.4)$$

$$\dot{\phi} = \sqrt{2b} F^{-1}, \quad (2.5)$$

where $\dot{} = \partial/\partial T$, $' = \partial/\partial R$, and γ , α and the scalar field strength b are real integration constants. A not immediately obvious consequence of this ansatz is the lack of an asymptotic AdS region (see below). The Ricci scalar of the metric (2.2) is

$$\mathcal{R} = 6\Lambda - \frac{2b^2}{F^4(T)}. \quad (2.6)$$

This may be used to show that this metric has a vanishing Cotton tensor and so is conformally flat, which also follows from the observation that it is conformal to a static metric with constant curvature spatial sections. For $F^2 = 0$ the conformal factor vanishes and the metric is singular. Solutions with a regular center correspond to the initial conditions $G(0) = 0$, $G'(0) = 1$, and thus necessarily $\alpha = +1$. In [10], these regular solutions were only presented as extensions of the Garfinkle solutions to which they reduce when the cosmological constant is switched off, but their properties, or those of the more general $\alpha \neq 1$ separable solutions, were not further discussed.

Here we briefly discuss the apparent horizon of these separable solutions, before focussing on the global structure of the $\Lambda < 0$, $\gamma > 0$ solutions. The location $r = r_{AH}$ of the apparent horizon of the metric (2.2) (with $r = F(T)G(R)$ the areal radius) is defined as the solution of the equation

$$g^{\mu\nu} \partial_\mu r \partial_\nu r = G^2 \left[-\frac{\dot{F}^2}{F^2} + \frac{G'^2}{G^2} \right] = 0. \quad (2.7)$$

On account of (2.3) and (2.4), this reduces to

$$\frac{\alpha}{G^2(R)} = \frac{b^2}{F^2(T)} + \Lambda F^2(T). \quad (2.8)$$

The slope of the apparent horizon is given by

$$\left. \frac{dR}{dT} \right|_{AH} = \frac{\dot{F}}{G'} \frac{dG}{dF} = \pm \frac{F}{G} \frac{dG}{dF} = \mp F^2 G^2 \frac{dG^{-2}}{dF^2} = \pm \frac{b^2 - \Lambda F^4}{b^2 + \Lambda F^4} \quad (2.9)$$

for $\alpha \neq 0$. The apparent horizon, if it exists, is for $\alpha \neq 0$ everywhere null for $b^2 = 0$ (vacuum) as well as for $\Lambda = 0$, and is null for $\Lambda \neq 0$, $F^2(T) = 0$. It is spacelike for $\alpha = 0$, as well as for $b^2 \neq 0$, $\alpha \neq 0$, and $\Lambda < 0$.

To understand how these solutions might describe gravitational collapse, it is important to know first how the background vacuum ($b^2 = 0$) spacetime is described by (2.2). For $\Lambda < 0$, solutions to (2.4) with $b^2 = 0$ exist only if $\gamma > 0$, which we assume in the following, choosing without loss of generality $\gamma = +1$. We first consider the case $\alpha < 0$. Putting $\Lambda = -l^{-2}$ and $\alpha = -M$, the solution of equations (2.3) and (2.4) is $G = \sqrt{M} \cosh R$, $F = l / \cosh T$, so that the metric (2.2) reads

$$ds^2 = \frac{l^2}{\cosh^2(T)} \left[-dT^2 + dR^2 + M \cosh^2(R) d\theta^2 \right]. \quad (2.10)$$

The spacetime is constant curvature and with a null bifurcate apparent horizon $R \pm T = 0$, so the metric (2.10) must be (locally) isometric to that of BTZ. Indeed, performing on the static BTZ metric

$$ds^2 = -(r^2/l^2 - M) dt^2 + \frac{dr^2}{r^2/l^2 - M} + r^2 d\theta^2 \quad (2.11)$$

the coordinate transformation

$$\frac{r}{l\sqrt{M}} = \frac{\cosh R}{\cosh T}, \quad \coth \left(\frac{\sqrt{M} t}{l} \right) = \left(\frac{\sinh R}{\sinh T} \right)^\epsilon, \quad (2.12)$$

with $\epsilon = \text{sgn}(r^2 - Ml^2)$, leads to the metric (2.10). Note however that the metric (2.10) covers only the domain

$$\tanh^2 \left(\frac{\sqrt{M} t}{l} \right) < \frac{r^2}{Ml^2} < \coth^2 \left(\frac{\sqrt{M} t}{l} \right) \quad (2.13)$$

of the maximally extended static BTZ spacetime. This domain, bounded by the coordinate singularity $T \rightarrow \infty$ of the metric (2.10), does not contain spacelike infinity $r \rightarrow \infty$, i.e. the metric (2.10) is not asymptotically AdS.

Similarly, the separable metric for $\alpha = 0$ ($G = e^R$)

$$ds^2 = \frac{l^2}{\cosh^2(T)} \left[-dT^2 + dR^2 + e^{2R} d\theta^2 \right] \quad (2.14)$$

is locally transformed into the Kruskal form of the BTZ vacuum ($M = 0$) metric [11]

$$ds^2 = \frac{l^2}{\rho^2} \left[-d\tau^2 + d\rho^2 + d\theta^2 \right] \quad (2.15)$$

($\rho = -l/r$) by the coordinate transformation

$$\tau = e^{-R} \sinh T, \quad \rho = -e^{-R} \cosh T. \quad (2.16)$$

Finally, the metric (2.2) with $\alpha > 0$ ($G = \sqrt{\alpha} \sinh R$) can be transformed into that of (a domain of) AdS_3 with a conical singularity (unless $\alpha = +1$) by the coordinate transformation

$$\frac{r}{l\sqrt{\alpha}} = \frac{\sinh R}{\cosh T}, \quad \cot \left(\frac{\sqrt{\alpha} t}{l} \right) = \frac{\cosh R}{\sinh T}. \quad (2.17)$$

Now we turn to the non-trivial solutions with a dynamical scalar field, corresponding to $b^2 \neq 0$. The metric (2.2) for these solutions may be written [10] in the explicit form¹

$$ds^2 = -\frac{l^2}{4} \frac{d\eta^2}{(\eta - \eta_-)(\eta_+ - \eta)} + \eta [dR^2 + G^2(R) d\theta^2], \quad (2.18)$$

with $\eta = F^2(T)$, and

$$\eta_{\pm} = \frac{l}{2} \left[l \pm \sqrt{l^2 + 4b^2} \right], \quad (2.19)$$

with $\eta_- < 0 \leq \eta \leq \eta_+$. The spacetime is time-symmetric around the turning point $\eta = \eta_+$, with two symmetrical past and future central spacelike singularities $\eta = 0$.

For $\alpha = -M < 0$ the spacetime is also, as in the vacuum case, left-right symmetric. Writing the equation of the apparent horizon (2.8) as $\eta = \eta_{AH}(R)$, we see that $bl < \eta_{AH}(R) \leq \eta_+$, the time-symmetric value $\eta_{AH} = \eta_+$ corresponding to the space-symmetric value $G = \sqrt{M}$. It follows that, as in the case of the BTZ black hole, the apparent horizon is bifurcate and divides the spacetime into two left and right exterior regions I and I' , and two future

¹Another, related, FRW explicit form is given in Appendix A.

and past interior regions II and II' . However, to the difference of the BTZ black hole, the apparent horizon is spacelike. Another remarkable difference is that, similarly to the case of its vacuum counterpart (2.10), the metric (2.18) is not asymptotically AdS. The corresponding spacetime is bounded by the spacelike singularity $F^2 = 0$, which for $b^2 \neq 0$ is a genuine curvature singularity. Near this singularity the metric (2.2) can be approximated, from (2.4), by

$$ds^2 \simeq F^2[-b^{-2} dF^2 + dR^2 + G^2(R) d\theta^2], \quad (2.20)$$

so that the two-dimensional sections are conformal to Minkowski₂, the two components of the singularity $\eta \equiv F^2 = 0$ constituting the boundary of the spacetime (Fig. 1a).

Because asymptotic infinity is only a point, this solution does not have an event horizon. It does have in common with the static BTZ black hole a bifurcate apparent horizon. Because of this topological similarity, we will refer to this dynamical solution as “black-hole like”. By analogy with the case of the BTZ black hole, we introduce the mass function

$$M(\eta_{AH}) \equiv \frac{r_{AH}^2}{l^2}. \quad (2.21)$$

In the case of the BTZ black hole ($b^2 = 0$), the apparent horizon coincides with the event horizon $r^2 = Ml^2$, so that $M(\eta_{AH}) = M$. For $b^2 \neq 0$,

$$M(\eta_{AH}) = \frac{M\eta_{AH}^2}{\eta_{AH}^2 - b^2 l^2} \geq \frac{M\eta_+}{l^2} > M. \quad (2.22)$$

We define the effective black hole mass as the minimum value of $M(\eta_{AH})$,

$$M_{eff} = \frac{M\eta_+}{l^2}, \quad (2.23)$$

which is attained at the moment of time symmetry. For small scalar field strengths, $b^2 \ll l^2$, $M_{eff} \simeq M(1 + b^2/l^2)$, with Mb^2/l^2 the scalar field contribution to the black hole mass, while for large scalar field strengths, $M_{eff} \simeq Mb/l$.

For $\alpha = 0$, $\eta_{AH} = bl < \eta_+$, so that the solution, given by (2.18) with $G = e^R$ has an apparent horizon with two disjoint components, dividing the spacetime into two future and past interior regions II and II' and a single exterior region I (Fig. 1b). The minimum value of the mass function $M(\eta_{AH}) = bl^{-1}e^{2R}$ is now zero. This solution corresponds to that discussed

in [8]. In this case, the coordinate transformation (2.16) transforms the separable metric (2.2) into the explicitly conformally flat metric [8]

$$ds^2 = \frac{F^2(T)}{\rho^2 - \tau^2} \left[-d\tau^2 + d\rho^2 + d\theta^2 \right] \quad (\tanh T = -\tau/\rho) \quad (2.24)$$

(an explicit form of the function $F^2(T)$ in terms of elliptic functions is given in [8]). In the limit $b^2 \rightarrow 0$ this reduces to the BTZ vacuum metric (2.15).

For $\alpha > 0$, $\eta_{AH} < bl < \eta_+$, and again there is an apparent horizon with two disjoint components. However the spacetime boundary has now a third component, the center $G = 0$ which is regular if $\alpha = +1$ (Fig. 1c).

3 The sub-threshold solution near the singularity

We wish to explore whether the $b^2 \neq 0$, $\alpha = 0$ solution, which lies at the threshold for $\alpha < 0$ scalar field black-hole like solutions and reduces to the BTZ vacuum for $b^2 \rightarrow 0$, presents the properties required of the critical solution for scalar field collapse in AdS_3 . To this end, we shall compare the behavior of the sub-threshold $\alpha > 0$ solution near the spacelike curvature singularity with that observed in the numerical simulations of [3]. Near the singularity, the cosmological constant can be neglected, so that the function $F(T)$ can be approximated by the solution of (2.4) with $\Lambda = 0$, $F = b \sinh T$, while the general solution of (2.3) can be written, up to a translation, as $G = e^R - (\alpha/4)e^{-R}$, and the family of near-threshold solutions near the singularity $T = 0$ can be written

$$\begin{aligned} ds_0^2 &= b^2 \sinh^2(T) \left[-dT^2 + dR^2 + \left(e^R - \frac{\alpha}{4} e^{-R} \right)^2 d\theta^2 \right], \\ \phi_0 &= \sqrt{2} \ln \tanh(-T/2) \end{aligned} \quad (3.1)$$

(we assume $T < 0$). The metric (3.1) can be put in the double null form

$$ds^2 = e^{2\sigma} du dv + r^2 d\theta^2 \quad (3.2)$$

by the coordinate transformation

$$u = b e^{R-T}, \quad v = b e^{R+T} \quad (3.3)$$

($u > v > 0$), leading to

$$ds_0^2 = \frac{(u-v)^2}{4} \left[b^2 \frac{dudv}{(uv)^2} + \left(1 - \frac{\beta^2}{4uv} \right)^2 d\theta^2 \right], \quad \phi_0 = \sqrt{2} \ln \left(\frac{u^{1/2} - v^{1/2}}{u^{1/2} + v^{1/2}} \right), \quad (3.4)$$

with $\beta^2 = \alpha b^2$.

A necessary condition for the $\alpha = 0$ solution $(\bar{\sigma}_0, \bar{r}_0, \bar{\phi}_0)$ to qualify as a possible critical solution is that it be self-similar. Let us first show that this is indeed the case. For $\Lambda = 0$ the solution of the Einstein equation $r_{uv} = 0$ can be gauge transformed to $r_0 = (u - v)/2$. Inserting the self-similar ansatz [5]

$$\phi = c \ln u + \psi(y), \quad (3.5)$$

with $y^2 = v/u$, into the scalar field equation

$$2r\phi_{,uv} + r_{,u}\phi_{,v} + r_{,v}\phi_{,u} = 0 \quad (3.6)$$

we obtain the general solution (up to an additive constant)

$$\phi = c \ln u + (c + d) \ln(1 - y) + (c - d) \ln(1 + y), \quad (3.7)$$

with c and d two independent constants². The assumption of a regular center at $r = 0$ ($y = \pm 1$) leads to Garfinkle's solutions with $c = \mp d$ [5], while the solution (3.4) for $\alpha = 0$ corresponds to $c = 0$, $d^2 = 2$.

For $\alpha > 0$ the spacetime structure is more obvious in the alternate double-null coordinates \bar{u}, \bar{v} defined by

$$\begin{aligned} \bar{u} &= u + \frac{\beta^2}{4} u^{-1} = \beta \cosh(\bar{R} - T), \\ \bar{v} &= v + \frac{\beta^2}{4} v^{-1} = \beta \cosh(\bar{R} + T), \end{aligned} \quad (3.8)$$

with $\bar{R} = R - (1/2) \ln(\alpha/4)$. These coordinates are such that $r = (\bar{u} - \bar{v})/2$, which vanishes either for $\bar{R} = 0$ (timelike center, corresponding to a conical singularity if $\alpha \neq 1$) or $T = 0$ (spacelike curvature singularity). In the sector $(\bar{R} > 0, T < 0)$ the apparent horizon is at $T = -\bar{R}$, corresponding to $\bar{v} = \beta$ ($v = \beta/2$), dividing the spacetime in two sectors *I* and *II*. Noting that $d\bar{v}/dv = 1 - e^{-2(R+T)}$ is negative in sector *I* (the sector of the center) and positive in sector *II* (the sector of the curvature singularity), we obtain the form of the metric

$$ds_0^2 = \mp \frac{\bar{u}\bar{v} \pm \sqrt{(\bar{u}^2 - \beta^2)(\bar{v}^2 - \beta^2)} - \beta^2}{2\alpha\sqrt{(\bar{u}^2 - \beta^2)(\bar{v}^2 - \beta^2)}} d\bar{u}d\bar{v} + \frac{(\bar{u} - \bar{v})^2}{4} d\theta^2, \quad (3.9)$$

with the upper sign in sector *I* and the lower sign in sector *II*.

²This solution was previously given in [10], Eq. (2.14).

For comparison with the numerical simulations, we define a time coordinate \hat{u} (noted t_c in [3] and u in [5]) as the null coordinate which coincides with the proper time of the central observer, measured from the curvature singularity. At the center, $\bar{v} = \bar{u}$, so that (3.9) reduces to

$$-d\hat{u}^2 = -\alpha^{-1} d\bar{u}^2, \quad (3.10)$$

leading to

$$\hat{u} = \alpha^{-1/2}(\beta - \bar{u}) = b[1 - \cosh(\bar{R} - T)]. \quad (3.11)$$

Putting $\hat{u} = -e^{-\hat{T}}$ and defining a relative radial coordinate \hat{R} by

$$\hat{R} = r e^{\hat{T}} = \frac{\alpha^{1/2}}{2} \frac{\bar{u} - \bar{v}}{\bar{u} - \beta}, \quad (3.12)$$

we show in Appendix B that, assuming \hat{T} large, the scalar field is given in sector I by

$$\phi = \sqrt{2} \left(-\frac{1}{2} \hat{T} + \ln \left[\frac{1 + \sqrt{1 - 2\alpha^{-1/2} \hat{R}}}{2} \right] + O(e^{-\hat{T}}) \right). \quad (3.13)$$

We note that, to this approximation, this coincides with Eq. (18) of [5] (where κ has been set to 4π) for the $n = 1$ Garfinkle solution, provided the radial coordinate R in Eq. (18) is replaced by $\alpha^{-1/2} \hat{R}$.

Following [3], we define the logarithmic coordinates

$$Z = \ln r = \ln \hat{R} - \hat{T}, \quad \hat{Z} = Z - \frac{1}{2} \ln \alpha, \quad (3.14)$$

and compute

$$\phi_{,ZZ}(Z, \hat{T}) = -\frac{1}{\sqrt{2}} \frac{e^{\hat{Z} + \hat{T}}}{(1 - 2e^{\hat{Z} + \hat{T}})^{3/2}} + O(e^{-\hat{T}}). \quad (3.15)$$

At a fixed \hat{T} , this quantity depends on Z only through the combination $Z - (1/2) \ln \alpha$, in accordance with Fig. 17 of [3], which shows to a good approximation that when a point particle of mass $1 - e^{-2A(0,0)}$ (a deficit angle $\omega = 2\pi(1 - e^{-A(0,0)})$) is introduced at the center, $\phi_{,ZZ}$ depends only on Z through the difference $Z - A(0,0)$. The fact that $\phi_{,ZZ}$ depends only on the sum $Z + \hat{T}$ is also in good agreement with Fig. 10 of [3].

We can also evaluate the Ricci scalar (2.6). From (B.7) this diverges at the origin as

$$\mathcal{R}(0) \simeq -\frac{1}{2} e^{2\hat{T}}, \quad (3.16)$$

in accordance with [3], where it was found that the value of the curvature scalar R at the origin diverges like $1/t_c^2$ as one approaches the accumulation point. The combination

$$r^2 \mathcal{R} \simeq -2\alpha \left(\frac{1 - e^{\hat{Z} + \hat{T}} - \sqrt{1 - 2e^{\hat{Z} + \hat{T}}}}{e^{\hat{Z} + \hat{T}}} \right)^2. \quad (3.17)$$

depends only on $Z + \hat{T}$ and goes to zero for $Z \rightarrow -\infty$ (T fixed) as $-(\alpha/2)e^{2(\hat{Z} + \hat{T})}$, in agreement with Fig. 12 of [3]. The mass aspect

$$M(R, T) = \frac{r^2}{l^2} - g^{\mu\nu} \partial_\mu r \partial_\nu r = \frac{b^2 G^2}{F^2} - \alpha \simeq -\frac{r^2 \mathcal{R}}{2} - \alpha \quad (3.18)$$

similarly depends only on $\hat{Z} + \hat{T}$ and goes to $-\alpha$ for $Z \rightarrow -\infty$ at T fixed, in agreement (in the case $\alpha = 1$) with Fig. 11 of [3].

We have already noted that, without the higher-order corrections, the scalar field of Eq. (3.13) is exactly that which would have been obtained by the self-similar ansatz of [5] for the parameter $n = 1$ if an angular deficit $\omega = 2\pi(1 - \alpha^{1/2})$ had been introduced in the metric ansatz. However for $\alpha = 1$ this does not agree with Fig. 2 of [5], which is well fitted by $n = 4$. To compare with Fig. 1 of [5], we can evaluate exactly

$$\left. \frac{\partial \phi_0}{\partial \hat{T}} \right|_{r=0} = \frac{\sqrt{2}}{\sinh T} \left. \frac{\partial T}{\partial \hat{T}} \right|_{r=0} = -\frac{1}{\sqrt{2}} \left[1 + \frac{e^{-\hat{T}}}{2b} \right]^{-1}, \quad (3.19)$$

which goes to the limit $-1/\sqrt{2}$ when $\hat{T} \rightarrow \infty$. In comparison, Fig. 1 of [5] for $\partial \phi_0 / \partial \hat{T}|_{r=0}$ shows a constant plateau at approximately -0.264 , corresponding to a value of the Garfinkle parameter $n = 4$, followed by an increase which might be consistent with the limiting value $-1/\sqrt{8\pi} \simeq -0.199$. However (after rescaling our scalar field by $1/\kappa$ with $\kappa = \sqrt{4\pi}$), our (3.19) decreases rather than increases towards this value.

4 Perturbations

Another property required of a critical solution is that it has only one unstable growing mode under linear perturbations. To study the linear perturbations of the $\alpha = 0$ solution, (2.18) with $G = e^R$, we follow the procedure described in [7]. Taking as independent variables u and $y = (v/u)^{1/2}$, the

Einstein and scalar field equations (2.1) may be rewritten as

$$2r_{uy} - u^{-1}(yr_{yy} + r_y) = 2\Lambda u y e^{2\sigma}, \quad (4.1)$$

$$4\sigma_{uy} - 2u^{-1}(y\sigma_{yy} + \sigma_y) = 2\Lambda u y e^{2\sigma} + \phi_y(-2\phi_u + u^{-1}y\phi_y), \quad (4.2)$$

$$yr_{yy} - r_y - 2y\sigma_y r_y = -ry\phi_y^2, \quad (4.3)$$

$$\begin{aligned} -r_{uu} + u^{-1}yr_{uy} - u^{-2}yr_y + 2\sigma_u r_u - u^{-1}y(\sigma_u r_y + \sigma_y r_u) = \\ = r(\phi_u^2 - u^{-1}y\phi_u\phi_y), \end{aligned} \quad (4.4)$$

$$r[2\phi_{uy} - u^{-1}(y\phi_{yy} + \phi_y)] + r_u\phi_y + r_y\phi_u - u^{-1}yr_y\phi_y = 0. \quad (4.5)$$

The $\alpha = 0$ solution is put in the double-null form (3.2) by the coordinate transformation (3.3). A generic solution (r, ϕ, σ) of the field equations is linearized around this background solution $(\bar{r}, \bar{\phi}, \bar{\sigma})$ as

$$\begin{aligned} r &= \bar{r}(u, y) + \varepsilon u^{1-k} f(y), \\ \phi &= \bar{\phi} + \varepsilon u^{-k} \sqrt{2} g(y), \\ \sigma &= \bar{\sigma} + \varepsilon u^{-k} h(y), \end{aligned} \quad (4.6)$$

with k a real constant. Modes with $\text{Re}(k) > 0$ grow as the singularity $u = 0$ is approached. To separate the linearized field equations, both the background solution and its linear perturbations are expanded in powers of the dimensionless quantity Λu^2 , so that to each order the linearized field equations reduce to ordinary differential equations in the variable y . These differential equations are solved iteratively, order by order. Here we shall only treat the problem to zeroth order, with the $\Lambda = 0$ solution (3.4) with $\beta = 0$ as background,

$$\begin{aligned} \bar{r}_0 &= u \frac{1 - y^2}{2}, \\ \bar{\phi}_0 &= \sqrt{2} \ln \frac{1 - y}{1 + y}, \\ \bar{\sigma}_0 &= \ln(b/2) - \ln u + \ln(1 - y^2) - 2 \ln y. \end{aligned} \quad (4.7)$$

Taking $F = b \sinh T$ in (2.24) leads to the alternate form of the truncated background metric

$$ds_0^2 = \frac{b^2 \tau^2}{(\rho^2 - \tau^2)^2} (-d\tau^2 + d\rho^2 + d\theta^2) \quad (0 < -\tau < \rho). \quad (4.8)$$

The structure of this spacetime may be visualized in the plane (x, τ) , where $x \equiv 1/\rho$. Noting that

$$x\tau = \frac{y^2 - 1}{y^2 + 1}, \quad (4.9)$$

we see that the curves $y = \text{constant}$ are hyperbolas bounded on one side by $x\tau = -1$ ($y = 0$), on the other side by $x\tau = 0$ ($y = 1$). The Penrose diagram of the truncated background is thus a triangle bounded by the regular timelike line $\rho = \infty$ ($R = \infty$), the singular spacelike line $\tau = 0$ ($T = 0$), both components of the boundary $y = 1$, and the null line $y = 0$ ($T = -\infty$). The intersections of the null line $y = 0$ with the timelike line $\rho = \infty$ and the spacelike line $\tau = 0$ correspond respectively to $u = \infty$ and $u = 0$. Actually, the null boundary $T = -\infty$, which is at infinite geodesic distance, is the limit $l \rightarrow \infty$ of the apparent horizon $F_{AH}^2 = bl$ of the spacetime (2.24), so that our truncated background is entirely contained in the future region II of the full background (Fig. 1b).

The linearization of (4.1), (4.5) and (4.3) with $\Lambda = 0$ around the truncated background (4.7) yields the differential equations

$$yf_0'' + (2k - 1)f_0' = 0, \quad (4.10)$$

$$y(1 - y^2)g_0'' + 2[k - (k + 1)y^2]g_0' - 2kyg_0 = \frac{4y}{1 - y^2}f_0' + \frac{4[k + (2 - k)y^2]}{(1 - y^2)^2}f_0, \quad (4.11)$$

$$2y(yh_0' - 2g_0') = -yf_0'' - \frac{3 + y^2}{1 - y^2}f_0' - \frac{8y}{(1 - y^2)^2}f_0, \quad (4.12)$$

and the linearization of (4.2) and (4.4) yields the extra (constraint) equations

$$yh_0'' + (2k + 1)h_0' - \frac{4(yg_0' + kg_0)}{1 - y^2} = 0, \quad (4.13)$$

$$\frac{y(1 - y^2)}{2}h_0' + kh_0 - 2kyg_0 = -(k - 1) \left[yf_0' + \left(k - 2 + \frac{2}{1 - y^2} \right) f_0 \right]. \quad (4.14)$$

To solve these equations one must enforce appropriate boundary conditions. The boundary of our truncated background (4.8) has two regular components. Our first boundary condition shall be that the perturbation be regular on the timelike boundary $\rho \rightarrow \infty$ ($y = 1$). On the other regular boundary component, $y = 0$, we shall enforce the condition that the perturbation be, as the background (4.7), analytic in y . The rationale for this condition is that, as we have seen, this null boundary of the truncated background is actually an apparent horizon of the full background, so that this analyticity condition ensures that the perturbation can be extended from region II to region I of the full background.

To discuss the solution of the system (4.10)-(4.12) we follow the procedure of [6]. The general solution of (4.10) is

$$f_0 = c_0 + c_1(1 - y^{2-2k}) \quad (4.15)$$

(for $k = 1$, a linearly independent solution is $c'_1 \log y$, which can be excluded by the analyticity condition). This is actually a pure gauge perturbation, generated from the background $\bar{r}_0 = u(1-y^2)/2$ by the gauge transformation

$$u \rightarrow u + 2\varepsilon(c_0 + c_1)u^{1-k}, \quad v \rightarrow v + 2\varepsilon c_1 v^{1-k}, \quad (4.16)$$

corresponding to

$$\delta y = \varepsilon u^{-k} y [c_1 y^{-2k} - (c_0 + c_1)]. \quad (4.17)$$

In the gauge $c_0 = c_1 = 0$, equation (4.11) is homogeneous. Its solution which is regular in $y = 1$ is

$$g_0 = c_2 F(k, 1/2, 1; 1 - y^2), \quad (4.18)$$

with F a hypergeometric function. To this we must add a particular solution of the inhomogeneous equation given by the gauge transformation (4.17) acting on the background solution

$$g_{(g)} = [c_0 + c_1(1 - y^{-2k})] \frac{2y}{1 - y^2}. \quad (4.19)$$

This is certainly not regular in $y = 1$ if $c_0 \neq 0$. Taking from now on $c_0 = 0$, and using an identity between hypergeometric functions, we obtain the solution of (4.11) in the gauge c_1

$$\begin{aligned} g_0 = & c_1 \frac{2y(1 - y^{-2k})}{1 - y^2} + \frac{c_2 \Gamma(1/2 - k)}{\sqrt{\pi} \Gamma(1 - k)} F(k, 1/2, k + 1/2; y^2) + \\ & + \frac{c_2 \Gamma(k - 1/2)}{\sqrt{\pi} \Gamma(k)} y^{1-2k} F(1 - k, 1/2, 3/2 - k; y^2), \end{aligned} \quad (4.20)$$

except if $k = (2p + 1)/2$ with p integer. In this last case the transformation between hypergeometric functions of $1 - y^2$ and of y^2 involves a non-analytic $\ln y$ term, so that the condition of analyticity at $y = 0$ dictates $c_2 = 0$, and the solution is pure gauge. Other non-integer values of k may similarly be eliminated because the third term is the product of an infinite even series by the non-analytic monomial y^{1-2k} .

There only remain the positive integer values $k = n$. For these values, the second term in (4.20) is identically zero, while the hypergeometric function

in the third term is a polynomial of order $2k - 2$. Expanding in powers of y , we obtain

$$g_0 = -2c_1 y^{1-2k} \left[1 + y^2 + \dots + y^{2k-2} \right] + c_2 y^{1-2k} \left[1 + \frac{1-k}{3-2k} y^2 + \dots + O(y^{2k-2}) \right]. \quad (4.21)$$

The leading divergence is compensated in the gauge $c_2 = 2c_1$. For $k = 1$, the compensation is exact, $g_0 = 0$ in this gauge, $f_0 = 0$ by virtue of (4.15), and $h_0 = 0$ from (4.12) and (4.14), so that the perturbation vanishes altogether. For $k > 1$ there remains

$$g_0 = -2c_1 y^{3-2k} \left[\frac{k-2}{2k-3} + \dots + O(y^{2k-4}) \right]. \quad (4.22)$$

For $k = 2$ the sum contains only one term, which vanishes identically, so that the analytic solution in the gauge $c_2 = 2c_1$ is $g_0 = 0$. For $k > 2$, the solution is not analytic.

It remains to determine the function $h_0(y)$. For $k = 2$, $f_0 = c_1(1 - y^{-2})$ which annihilates the right-hand sides of (4.11), (4.12) and (4.14). Eq. (4.12) with $g_0 = 0$ then leads to $h'_0 = 0$, and finally $h_0 = 0$ from (4.14). The solution for $k = 2$ is thus

$$f_0 = c_1(1 - y^{-2}), \quad g_0 = 0, \quad h_0 = 0. \quad (4.23)$$

The resulting linearized solution (4.6) is identical, with the correspondence $\varepsilon c_1 = \alpha b^2/8$, to the exact near-threshold solution (3.4) with $\alpha \neq 0$.

5 Discussion

We have proposed an alternative scenario for circularly symmetric scalar field collapse in AdS_3 , in which the critical solution is self-similar but, instead of having a regular timelike center, is centerless. This scenario is motivated by the observed independence of the near-critical regime on the angle deficit of the spacetime. To illustrate this scenario, we have presented a family of exact, dynamical solutions to the three-dimensional Einstein-scalar field equations with negative cosmological constant $\Lambda = -l^{-2}$. These solutions depend on two parameters, a mass parameter $M = -\alpha$, and a parameter b measuring the scalar field strength. For $b = 0$ these solutions reduce locally to the static BTZ family of vacuum solutions. For $b \neq 0$ and $\alpha < 0$, the dynamical solution is black-hole like, with a global structure topologically similar to that of the BTZ black hole, and a finite effective mass. This

effective mass vanishes for the $\alpha = 0$ solution, which is self-similar and centerless. The solutions below the black hole threshold, $\alpha > 0$, have a deficit angle $2\pi(1 - \alpha)$. We have discussed the near-singularity behavior of these solutions and shown that it agrees qualitatively with that observed in numerical simulations of subcritical collapse. Finally, we have analyzed in the $\Lambda = 0$ approximation the linear perturbations of the centerless threshold solution, $\alpha = 0$. Assuming reasonable boundary conditions, we found that it has only one unstable growing mode, as expected for the critical solution.

The realization of the no-center scenario discussed here presents some shortcomings. A first problem is that our candidate critical and subcritical solutions are not asymptotically AdS, while one would expect the true critical solution for scalar field collapse in AdS_3 to be asymptotically AdS. A second, perhaps related, problem concerns the quantitative comparison of these solutions, which behave like the $n = 1$ Garfinkle solution near the singularity, with the numerical simulations, which are best fit by the $n = 4$ Garfinkle solution [5]. Another problem is that subcritical solutions should be regular, and the critical solution should not have trapped surfaces, while our threshold and sub-threshold solutions both present apparent horizons and spacelike singularities. However this is also the case for the Garfinkle critical solutions (the corresponding subcritical solutions are not known in closed form, and therefore one cannot exclude that the same features would be present there as well), which also present an apparent horizon and, for n odd, a spacelike singularity [12]. As in the Garfinkle case [5], one should not expect the numerical critical solution to approach our exact self-similar critical solution outside the past light cone of the singularity $u = 0$. A fortiori we would not expect our exact $\alpha = 1$ subthreshold solutions, which are regular at the origin but not asymptotically AdS, to reproduce all the features of the observed near-critical behavior.

A last problem concerns the value of the critical exponent γ . This is defined by the scaling relation of the effective black hole mass $M_{eff} \propto |p - p^*|^{2\gamma}$, and is related to the mode eigenvalue k by $\gamma = 1/k$. Our family of near-critical solutions depends on the parameter $\alpha = -M$ with the critical value $\alpha^* = 0$, so that (2.23) leads to the value $\gamma = 1/2$. This agrees with the value obtained in our linear perturbation analysis, $k = 2$, corresponding to a critical exponent $\gamma = 1/2$. However this value disagrees with the numerical evaluations of [3] from maximum central curvature scaling, leading to $\gamma = 1.20 \pm 0.05$, and of [4] from apparent horizon mass scaling, which yield $\gamma \simeq 0.81$.

An interesting collateral result of our investigation concerns the existence of black holes in three-dimensional spacetime. It is often taken for granted

that the only black hole solutions to the three-dimensional Einstein-scalar field system are the BTZ black holes, with negative cosmological constant and vanishing scalar field. Actually, two non-trivial counterexamples with zero cosmological constant are already known: the cold black holes of [13] with a negative gravitational constant; and (in the case of a positive gravitational constant) the Garfinkle solutions with n odd. In the present work, we have presented a new dynamical scalar-field black-hole like solution, with a spacelike singularity hidden behind a spacelike apparent horizon, corresponding to the over-threshold case $\Lambda < 0$ and $\alpha \leq 0$, with $M_{eff} \geq 0$.

In conclusion, we have presented a scenario for critical scalar field collapse in AdS_3 alternative to that of [5] which, despite a number of shortcomings, has the advantage of explaining black hole formation, and the independence of the critical behavior on the angle deficit of the spacetime. Perhaps there is a (yet unknown) family of asymptotically AdS solutions which reduces to the BTZ family when the scalar field is switched off, with a self-similar critical solution devoid of a timelike center, and with subcritical solutions which would behave near the singularity like the $n = 4$ Garfinkle solutions.

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Appendix A: FRW form of the separable solution

The solution (2.2)-(2.5) may also be written in the Friedmann-Robertson-Walker form

$$ds^2 = -d\bar{t}^2 + \frac{l^2}{2} \left[1 + \nu \cos \left(\frac{2\bar{t}}{l} \right) \right] \left[\frac{d\bar{r}^2}{\alpha + \bar{r}^2} + \bar{r}^2 d\theta^2 \right], \quad (\text{A.1})$$

$$\phi = \frac{1}{\sqrt{2}} \ln \left[\frac{(1 + \nu) l/2b + \tan(\bar{t}/l)}{(1 + \nu) l/2b - \tan(\bar{t}/l)} \right] \quad \left(\nu = \sqrt{1 + 4b^2/l^2} \right), \quad (\text{A.2})$$

where $d\bar{t} = F(T)dT$ and $\bar{r} = G(R)$. In the weak-amplitude limit (no back-reaction) $b^2 \ll l^2$, the metric (A.1) reduces to the BTZ cosmology

$$ds^2 = -d\bar{t}^2 + l^2 \cos^2 \left(\frac{\bar{t}}{l} \right) \left[\frac{d\bar{r}^2}{\alpha + \bar{r}^2} + \bar{r}^2 d\theta^2 \right], \quad (\text{A.3})$$

and the scalar field (A.2) reduces to the solution of the wave equation

$$\phi = \frac{\sqrt{2}b}{l} \tan\left(\frac{\bar{t}}{\bar{l}}\right) = \frac{\sqrt{2}b}{l} \sinh T = \frac{\sqrt{2}b}{l} \left(\frac{r^2 + \alpha l^2}{\alpha l^2 \cot^2(\sqrt{\alpha}t/l) - r^2} \right)^{1/2} \quad (\text{A.4})$$

over the BTZ background (A.3), (2.10) or (2.11) (with $M = -\alpha$), respectively. The scalar field (A.4) diverges at the coordinate singularity $T \rightarrow \infty$ of (2.10) (or the singularity $\bar{t} = \pi l/2$ of (A.3)).

Noting that in the coordinates of (A.1) $d\hat{u} = -d\bar{t}$ at r fixed, we can use (A.2) to evaluate directly $\partial\phi/\partial\hat{T}|_{r=0}$ without the approximation $\Lambda = 0$, leading to

$$\left. \frac{\partial\phi}{\partial\hat{T}} \right|_{r=0} = -\frac{1}{\sqrt{2}} \frac{\hat{u}}{\sin\hat{u}[\cos\hat{u} - (l/2b)\sin\hat{u}]}, \quad (\text{A.5})$$

with $\hat{u} = -e^{-\hat{T}}$. This reduces to (3.19) in the limit $l \rightarrow \infty$, but has a rather different behavior when $b/l = O(1)$, and for large b/l increases towards the limit $-1/\sqrt{2}$ when $\hat{T} \rightarrow \infty$.

Appendix B: Near-singularity behavior of the scalar field (3.1)

We first evaluate the relative radial coordinate \hat{R} of (3.12) in terms of \hat{T} and \bar{R} . The numerator is

$$\begin{aligned} \bar{u} - \bar{v} &= \beta[\cosh(\bar{R} - T) - \cosh(\bar{R} + T)] \\ &= 2\beta \sinh \bar{R} [\sinh(\bar{R} - T) \cosh \bar{R} - \cosh(\bar{R} - T) \sinh \bar{R}] \\ &= \frac{8\beta\nu(\mu - \nu)(1 - \mu\nu)}{(1 - \mu^2)(1 - \nu^2)^2}, \end{aligned} \quad (\text{B.1})$$

where we have put $\mu = \tanh[(\bar{R} - T)/2]$, $\nu = \tanh[\bar{R}/2]$. We note that

$$e^{-\hat{T}} = b[\cosh(\bar{R} - T) - 1] = \frac{2b\mu^2}{1 - \mu^2}, \quad (\text{B.2})$$

so that \hat{T} large means $\mu^2 \ll 1$. Also, in the region I, $\bar{R} < -T$, implying $\bar{R} < (\bar{R} - T)/2$, hence $\nu < \mu/2$, so that $\mu\nu$ and ν^2 are also small. So we can approximate (B.1) by

$$\bar{u} - \bar{v} \simeq 8\beta\nu(\mu - \nu). \quad (\text{B.3})$$

Similarly, $\bar{u} - \beta = \alpha^{1/2}e^{-\hat{T}} \simeq 2\beta\mu^2$, so that

$$\hat{R} \simeq \frac{2\alpha^{1/2}\nu(\mu - \nu)}{\mu^2} = 2\alpha^{1/2}x(1 - x), \quad (\text{B.4})$$

with $x = \nu/\mu < 1/2$. This can be inverted to yield

$$x = \frac{1 - \sqrt{1 - 2\alpha^{-1/2}\hat{R}}}{2}. \quad (\text{B.5})$$

Next we compute

$$\begin{aligned} \phi &= \sqrt{2} \ln \tanh(-T/2) = \sqrt{2} \ln \tanh \left[(\bar{R} - T)/2 - T/2 \right] \\ &= \sqrt{2} \ln \left(\frac{\mu - \nu}{1 - \mu\nu} \right) \simeq \sqrt{2} [\ln \mu + \ln(1 - x)]. \end{aligned} \quad (\text{B.6})$$

After using (B.5), and subtracting a constant from the scalar field, we obtain Eq. (3.13), where the neglected terms are of order $O(\mu^2) = O(e^{-\hat{T}})$.

From (2.6) the Ricci scalar is

$$\mathcal{R} = -6l^{-2} - \frac{2}{b^2 \sinh^4 T} \simeq -\frac{1}{8b^2(\mu - \nu)^2} \simeq -\frac{e^{2\hat{T}}}{2(1 - x)^4}, \quad (\text{B.7})$$

leading to

$$r^2 \mathcal{R} = -2\alpha \frac{x^2}{(1 - x)^2}. \quad (\text{B.8})$$

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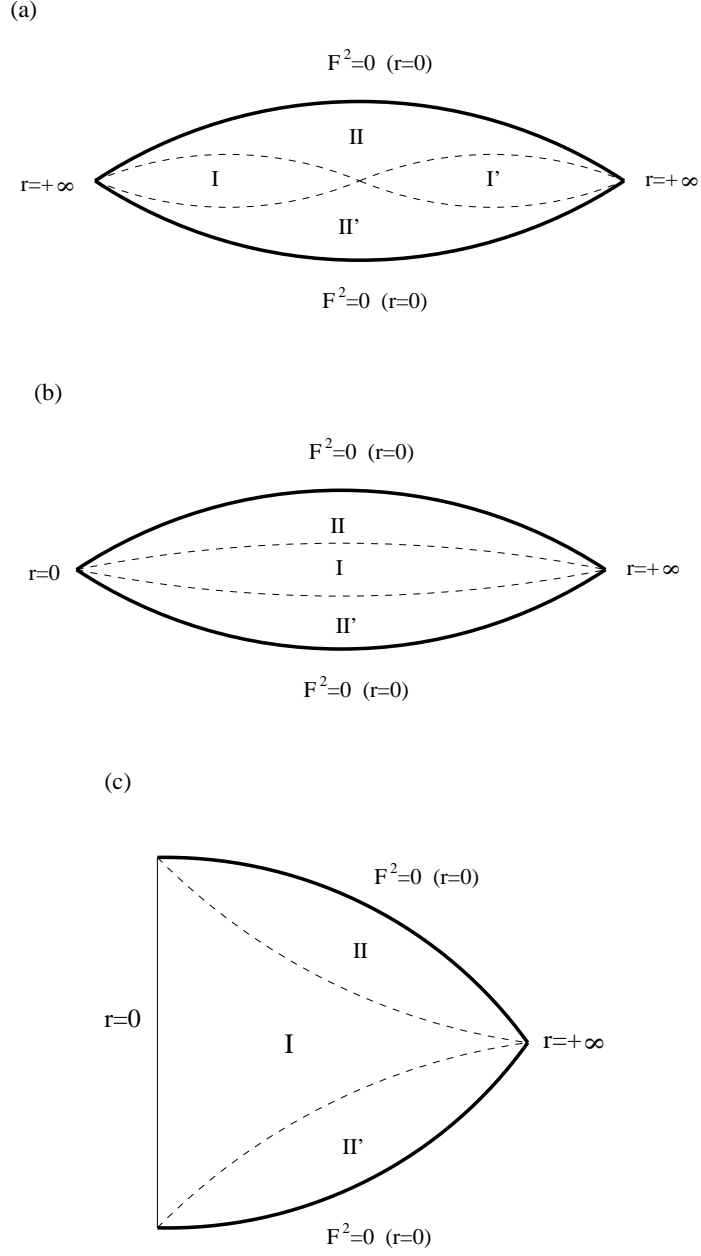


Figure 1: Penrose diagram for the solution (2.18)
(a) for $\alpha < 0$, (b) for $\alpha = 0$, (c) for $\alpha > 0$. The curvature singularity is shown as a heavy line, the apparent horizon as a broken line.